**THEOREM 0.1.** *Let $f$ be a function defined on a rectangle*

$$R := \{(t, y) : |t - t_0| < a,\ |y - y_0| < b\},\ a,\ b > 0.$$

*such that*
*(i) $f$ is bounded on $R$, i.e., there exists $M > 0$ such that $|f(t, y)| \leq M$ for all $(t, y) \in R$;*
*(ii) $f$ is Lipschitz continuous in variable $y$, uniformly in $t$ i.e., there exists $L > 0$ such that*

$$|f(t, y_1) - f(t, y_2)| \leq L|y_1 - y_2| \text{ for all } |t - t_0| < a.$$

*Then for $\varepsilon < \min\{\frac{1}{L},\ \frac{b}{M}\}$, there exists a unique function $y : (t_0 - \varepsilon, t_0 + \varepsilon) \to R$ which is a solution to the initial value problem*

$$\left.\begin{array}{rcl} \frac{dy}{dt} & = & f(t, y(t)) \\ y(t_0) & = & y_0. \end{array}\right\} \tag{0.1}$$

*The uniqueness is in the sense that if $\tilde{y}$ defined on an interval $(t_0 - \eta, t_0 + \eta)$ is another solution of IVP (0.1) then*

$$\tilde{y}(x) = y(x) \text{ for all } x \in (t_0 - \eta, t_0 + \eta) \cap (t_0 - \varepsilon, t_0 + \varepsilon).$$

*Proof.* **Step I:** A function $y : (t_0 - \varepsilon, t_0 + \varepsilon) \to I\!R$ is a solution to the differential equation (0.1) iff it is a solution of the integral equation

$$y(t) = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds. \tag{0.2}$$

For if $y$ is a solution of (0.1) then integrating from $t_0$ to $t$ we get

$$\int_{t_0}^{t} \frac{dy}{dt} = \int_{t_0}^{t} f(s, y(s))\, ds$$

and hence

$$y(t) - y(t_0) = \int_{t_0}^{t} f(s, y(s))\, ds$$

which is (0.2). Conversely, let $y$ solves the integral equation (0.2). Then $y(t_0) = y_0$ and by fundamental theorem of integral calculus, $y$ is differentable and its derivative is given by

$$\frac{dy}{dt} = f(t, y(t)).$$

**Step II: Existence of solution- Picard's iteration scheme**
Define functions $y_0(t) = y_0$ for all $t$, $y_1$ as

$$y_1(t) = y_0 + \int_{t_0}^{t} f(s, y_0(s))\, ds = y_0 + \int_{t_0}^{t} f(s, y_0)\, ds,$$

and functions $y_k$ which are iteratively defined as

$$y_k(t) = y_0 + \int_{t_0}^{t} f(s, y_{k-1}(s))\, ds. \tag{0.3}$$

By Fundamental theorem of integral calculus, the functions $\{y_k\}$ are differentiable.

**Claim 1:** There exists $\varepsilon_1 > 0$, independent of $k$, such that $(t, y_k(t)) \in R$ for all $t \in (t_0 - \varepsilon_1, t_0 + \varepsilon_1)$.
*Proof of Claim 1:* We have

$$\begin{aligned} |y_k(t) - y_0| &= |\int_{t_0}^t f(s, y_{k-1}(s))\, ds| \\ &\leq \int_{t_0}^t |f(s, y_{k-1}(s))\, ds| \\ &\leq M|t - t_0|. \end{aligned}$$

Thus if we choose $\varepsilon_1 < \frac{b}{M}$, then $|y_k(t) - y_0| < b$ for all $t \in (t_0 - \varepsilon_1, t_0 + \varepsilon_1)$ and the claim follows.

**Claim 2:(Convergence of successive approximations)** The sequence $\{y_k\}$ converges uniformly to a function $y$ in an interval $(t_0 - \varepsilon, t_0 + \varepsilon)$ for $0 < \varepsilon \leq \varepsilon_1$.
*Proof of Claim 2:* Note that

$$|y_1(t) - y_0(t)| \leq \int_{t_0}^t (f(s, y_0(s))\, ds \leq M(t - t_0)$$

and that

$$\begin{aligned} |y_2(t) - y_1(t)| &= |\int_{t_0}^t \{f(s, y_1(s)) - f(s, y_0(s))\}\, ds| \\ &\leq \int_{t_0}^t |f(s, y_1(s)) - f(s, y_0(s))|\, ds \\ &\leq L\int_{t_0}^t |y_1(s) - y_0(s)|\, ds \\ &\leq LM\int_{t_0}^t (s - t_0)\, ds \\ &= LM\frac{(t - t_0)^2}{2}. \end{aligned}$$

**Ex.** Prove by induction

$$|y_{n+1}(t) - y_n(t)| \leq M\frac{L^n(t - t_0)^{n+1}}{(n+1)!}$$

Thus, for $m \geq n$ we have

$$\begin{aligned} |y_m(t) - y_n(t)| &\leq |y_m(t) - y_{m-1}(t)| + |y_{m-1}(t) - y_{m-2}(t)| + \ldots + |y_{n+1}(t) - y_n(t)| \\ &\leq \frac{M}{L}\sum_{k=n}^m \frac{[L(t - t_0)]^k}{k!} \\ &= \frac{M}{L}(S_m - S_{n-1}) \qquad (0.4) \end{aligned}$$

where $S_n = \sum_{k=0}^{n} \frac{[L(t-t_0)]^k}{k!}$ is the $n$th partial sum of the exponential series $e^{L(t-t_0)}$ which converges for all values of $(t-t_0)$ and in particular for $|t-t_0| \le a$. Note that

$$\begin{aligned} y_n(t) &= (y_n(t) - y_{n-1}(t)) + (y_{n-1}(t) - y_{n-2}(t)) + \ldots + (y_1(t) - y_0(t)) + y_0(t) \\ &= y_0(t) + \sum_{k=1}^{n} (y_k(t) - y_{k-1}(t)) \end{aligned}$$

and (0.4) shows that the partial sums of the series $y_0(t) + \sum_{k=1}^{\infty} (y_k(t) - y_{k-1}(t))$ is dominated by the partial sums of the series for $e^{L(t-t_0)}$ for all $t$. Thus, the series $y_0(t) + \sum_{k=1}^{\infty} (y_k(t) - y_{k-1}(t))$ converges absolutely for $|t-t_0| < \varepsilon_1$ and converges to a function denoted by $y(t)$ for $|t-t_0| < \varepsilon_1$.

Thus $y_n(t) \to y(t)$ for $|t-t_0| < \varepsilon_1$ pointwise. This convergence is uniform since,

$$\begin{aligned} |y_n(t) - y(t)| &= |\sum_{k=n+1}^{\infty} y_k(t) - y_{k-1}(t))| \\ &\le \frac{M}{L} \sum_{k=n+1}^{\infty} \frac{(L|t-t_0|)^k}{k!} \\ &\le ML \sum_{k=n+1}^{\infty} \frac{(La)^k}{k!} = ML|e^{aL} - T_n| \end{aligned}$$

where $T_n = \sum_{k=0}^{n} \frac{[La]^k}{k!}$ is the $n$th partial sum of the exponential series $e^{aL}$. Since $|e^{aL} - T_n| \to 0$ as $n \to \infty$, independent of $t$, it follows that $y_n$ converges uniformly to $y$ on $(t_0 - \varepsilon_1, t_0 + \varepsilon_1)$

**Claim 3:** $y$ is a solution of the IVP.
*Proof of Claim 3:* Since $y_k(t_0) = y_0$ for all $k$, taking limit as $k \to \infty$ we get $y(t_0) = y_0$. Furthermore, observe that the sequence of functions $\{f_k\}_k$ where $f_k(s) = f(s, y_k(s))$ is uniformly convergent. To see this, consider

$$|f_m(s) - f_n(s)| = |f(s, y_m(s)) - f(s, y_n(s))| \le L|y_m(s) - y_n(s)|.$$

Since $\{y_k\}$ is uniformly convergent for $|t-t_0| < \varepsilon$ and hence uniformly Cauchy sequence, it follows that $\{f_k\}_k$ is uniformly Cauchy sequence and hence is uniformly convergent for $|t-t_0| < \varepsilon$ . Moreover,

$$|f_m(s) - f(s, y(s))| = |f(s, y_m(s)) - f(s, y(s))| \le L|y_m(s) - y(s)|$$

implies that $f_m(t) \to f(t, y(t)$ uniformly for $|t-t_0| < \varepsilon$. Taking limit as $k \to \infty$ in (0.3) we get

$$y(t) = y_0 + \lim_{k\to\infty} \int_{t_0}^{t} f(s, y_k(s))\, ds = y_0 + \int_{t_0}^{t} \lim_{k\to\infty} f(s, y_k(s))\, ds = y_0 + \int_{t_0}^{t} f(s, y(s))\, ds.$$

Thus $y(t)$ is solution of the IVP for $|t - t_0| < \varepsilon$ where $\varepsilon < \min\{a, \frac{b}{M}\}$.

**Remark:** Note that for the existence of solution it suffices to choose $\varepsilon < \frac{b}{M}$, however, the proof of uniqueness requires $\varepsilon < \min\{\frac{1}{L}, \frac{b}{M}\}$ as can be seen below-

**Step III: Uniqueness**
Let $y$ defined on $(t_0-\varepsilon, t_0+\varepsilon)$ for some $0 < \varepsilon < \min\{\frac{1}{L}, \frac{b}{M}\}$ as obtained above and $\tilde{y}$ defined on $(t_0-\delta, t_0+\delta)$ for some $\delta > 0$ be solutions of the IVP (0.1). Let $I = (t_0-\varepsilon, t_0+\varepsilon) \cap (t_0-\delta, t_0+\delta)$ and suppose there exists $t_1 \in I$ such that $y(t_1) \neq \tilde{y}(t_1)$. Thus $\sup_{s\in I} |y(s) - \tilde{y}(s)| = m_0$(say) is strictly positive. Since both $y$ and $\tilde{y}$ are solutions of the IVP and hence of the integral equation (0.3) we have

$$\begin{aligned} |y(t) - \tilde{y}(t)| &= \left| \int_{t_0}^{t} (f(s, y(s)) - f(s, \tilde{y}(s)))\, ds \right| \\ &\leq \int_{t_0}^{t} |f(s, y(s)) - f(s, \tilde{y}(s))|\, ds \\ &\leq L \int_{t_0}^{t} |y(s) - \tilde{y}(s)|\, ds \\ &\leq L|t - t_0| \sup_{s\in I} |y(s) - \tilde{y}(s)| \\ &\leq L\varepsilon m_0. \end{aligned}$$

Taking supremum on l.h.s. as $t$ varies in $I$ we get $m_0 \leq L\varepsilon m_0$, i.e., $1 \leq L\varepsilon$. But we chose $\varepsilon$ such that $\varepsilon < \frac{1}{L}$, thus we obtain a contradiction. Hence $m_0 = 0$ and $y \equiv \tilde{y}$ on $I$.