Theorem 0.1. *(Existence of ε-approximate solution) Let f be a continuous function defined on the rectangle*

$$R := \{(t, y) : |t - t_0| \leq a, \; |y - y_0| \leq b\}.$$

Given $\varepsilon > 0$ there exists an ε-approximate solution φ of the initial value problem

$$\left.\begin{array}{rcl} \frac{dy}{dt} & = & f(t, y(t)) \\ y(t_0) & = & y_0 \end{array}\right\} \tag{0.1}$$

defined on an interval $I = (t_0, \eta, t_0 + \eta)$ such that

$$|\varphi(t) - \varphi(s)| < M|t - s| \quad \textit{for all} \quad s, t \in I \tag{0.2}$$

where $M = \sup_R |f(t, y)|$, i.e., φ is Lipshcitz continuous on I.

Proof. Recall that φ is said to be an ε- approximate solution of (0.1) on an interval I if
(i) φ is continuous on I with $(t, \varphi(t)) \in R$;
(ii) φ is C^1 on I except possibly on a finite set of points S on I, where φ' may be discontinuous;
(iii) $\varphi'(t) - f(t, \varphi(t))| \leq \varepsilon$ on $I \setminus S$. We will construct a piecewise linear ε approximate solution. First, the solution will be constructed to the right of t_0.
Step I: Draw a line segment

$$l(s) = y_0 + f(t_0, y_0)(t - t_0)$$

through the point (t_0, y_0). We see that $l(s) \in R$ if

$$|s - t_0| \leq a \tag{0.3}$$

and

$$|l(s) - y_0| = |f(t_0, y_0)||t - t_0| \leq M|t - t_0| \leq M\delta \leq b$$

if we choose $\eta \leq \min\{a, \frac{b}{M}\}$. Now consider the interval $I = [t_0 - \eta, t_0 + \eta]$. Since R closed and bounded set and f is continuous on R and thus uniformly continuous. Hence given $\varepsilon > 0$, there exists $\delta_1 > 0$ such that

$$|f(t, y) - f(s, x)| < \varepsilon \quad \text{whenever} \quad |t - s| < \delta_1, |y - x| < \delta_1. \tag{0.4}$$

Let

$$\delta < \min\{\delta_1, \frac{\delta_1}{M}\} \tag{0.5}$$

Divide the interval $[t_0, t_0 + \eta]$ into subintervals of length less than δ by choosing points

$$t_0 < t_1 < t_2 < \ldots < t_n = t_0 + \eta$$

such that $|t_i - t_{i-1}| < \delta$ for each $i = 1, \ldots, n$. Define

$$\begin{aligned} \varphi_1(s) &= l(s) = y_0 + f(t_0, y_0)(t - t_0) \quad \text{for} \quad t_0 \le s \le t_1 && (0.6) \\ \varphi_2(s) &= y_1 + f(t_1, y_1)(t - t_1) \quad \text{for} \quad s \in [t_1, t_2] \quad \text{where} \quad y_1 = \varphi_1(t_1) && (0.7) \end{aligned}$$

In general, denote $y_{i-1} = \varphi_{i-1}(t_{i-1})$ and define

$$\varphi_i(s) = y_{i-1} + f(t_{i-1}, y_{i-1})(t - t_{i-1}) \quad \text{for} \quad s \in [t_{i-1}, t_i] \quad \text{for} \quad i = 1, \ldots, n. \tag{0.8}$$

Let

$$\varphi(s) = \varphi_i(s) \quad \text{for} \quad s \in [t_{i-1}.t_i], \quad 1 \le i \le n. \tag{0.9}$$

Step II φ is an ε- approximate solution of (0.1). Let $s, t \in I$ and $s \in [t_{i-1}, t_i]$, $t \in [t_{j-1}, t_j]$ with $i < j$.

$$\begin{aligned} & |\varphi(t) - \varphi(s)| \\ \le\ & |\varphi(t) - \varphi(t_{j-1})| + |\varphi(t_{j-1}) - \varphi(t_{j-2})| + \ldots + |\varphi(t_i) - \varphi(s)| \\ \le\ & |f(t_{j-1}, y_{j-1})|(t - t_{j-1}) + |f(t_{j-2}, y_{j-2})|(t_{j-1} - t_{j-2}) + \ldots + |f(t_{i-1}, y_{i-1})|(t_i - s) \\ \le\ & M(t - s) && (0.10) \end{aligned}$$

In particular, φ is continuous. Also, from definition since $\varphi = \varphi_i$ on $[t_{i-1}, t_i]$, φ is C^1 on $I \setminus S$ where $S = \{t_1, t_2, \ldots, t_n = t_0 - \eta\}$. From (0.10) and since $\eta < \frac{b}{M}$ it follows that

$$(t, \varphi(t)) \in R \quad \text{for all} \quad t \in I.$$

For $s \in I$, say $s \in [t_{i-1}, t_i]$, $\varphi'(s) = \varphi_i'(s) = f(t_{i-1}, y_{i-1}) = f(t_{i-1}, \varphi(t_{i-1}))$ and hence

$$|\varphi'(s) - f(s, \varphi(s))| = |f(t_{i-1}, \varphi(t_{i-1})) - f(s, \varphi_i(s))| \le \varepsilon \tag{0.11}$$

since

$$|\varphi_i(s) - \varphi(t_{i-1})| \le M|s - t_{i-1}| \le \delta_1$$

Similarly, φ can be extended to the left of t_0. □

Lemma 0.2. *(Ascoli's lemma) Let I be a bounded interval and $\mathcal{F} = \{f\}$ be an infinite, uniformly bounded, equicontinuous set of functions. Then $\mathcal{F}$ contains a sequence $\{f_n\}$, $n = 1, 2, 3, \ldots$ which is uniformly convergent on I.*

Proof: List the rationals in $\mathbb{Q}$ as $\{r_k\}$, $k = 1, 2, 3, \ldots$. Let $A_1 = \{f(r_1) : f \in \mathcal{F}, k = 1, 2, 3, \ldots\}$. Since $\mathcal{F}$ is uniformly bounded, there exists $M > 0$ such that

$$|f(x)| \le M \quad \text{for all} \quad x \in I, \quad \text{for all} \quad f \in \mathcal{F}. \tag{0.12}$$

Then A_1 is an infinite set of real numbers bounded in I and hence has a convergent subsequence, say $f_{11}(r_1), f_{21}(r_1), f_{31}(r_1), \ldots f_{k1}(r_1) \ldots$. Let

$$\mathcal{F}_1 := \{f_{k1}\}_{k \in I\!N}.$$

Again, $A_2 := \{fk1(r_2) : k = 1, 2, 3, \ldots\}$ is a bounded set of real numbers and hence has a convergent subsequence, $\{f_{k2}\}_{k \in I\!N}$. Let

$$\mathcal{F}_2 := \{f_{k2}\}_{k \in I\!N} \subset \mathcal{F}_1.$$

Continuing thus we obtain sets

$$\mathcal{F}_j := \{f_{kj}\}_{k \in I\!N} \subset \mathcal{F}_{j-1} \subset \mathcal{F}_{j-2} \subset \ldots \subset \mathcal{F}_2 \subset \mathcal{F}_1$$

with the property that

$$\{f_{kj}(r_i)\} \text{ is convergent for all } \quad i \leq j. \tag{0.13}$$

Now define the sequence of functions

$$f_k = f_{kk}, \ k = 1, 2, 3, \ldots. \tag{0.14}$$

Claim: $\{f_k\}_{k \in I\!N}$ is uniformly convergent on I.
We show that $\{f_k\}$ is uniformly Cauchy sequence. Since $\mathcal{F}$ is equicontinuous, given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$|x - y| < \delta \implies |f(x) - f(y)| < \varepsilon \quad \text{for all } f \in \mathcal{F}. \tag{0.15}$$

Let $x \in I$ and $r_j \in \mathbb{Q}$ such that

$$|x - r_j| < \delta.$$

Then

$$|f_k(x) - f_k(r_j)| < \varepsilon.$$

Also, since $\{f_k(r_j)\}_k = \{f_{kk}(r_j)\}_k$ is convergent for all $k \geq j$, given $\varepsilon > 0$ there exists n_0 such that for all $m, \ n \geq n_0$,

$$|f_m(r_j) - f_n(r_j)| < \varepsilon.$$

Thus, for $m, \ n \geq n_0$

$$\begin{aligned} |f_n(x) - f_m(x)| &= |f_n(x) - f_n(r_j) + f_n(r_j) - f_m(r_j) + f_m(r_j) - f_m(x)| \\ &\leq |f_n(x) - f_n(r_j)| + |f_n(r_j) - f_m(r_j)| + |f_m(r_j) - f_m(x)| \\ &< \varepsilon + \varepsilon + \varepsilon = 3\varepsilon. \end{aligned}$$

□

Theorem 0.3. (Cauchy-Peano Existence theorem:) *If f is a continuous function on a rectangle R, then there exists a solution $\varphi \in C^1$ of (0.1) on $|t - t_0| \leq \eta$ for which $\varphi(t_0) = y_0$.*

Proof. Let $\varepsilon_n > 0$ denote a decreasing of positive numbers such that $\varepsilon_n \to 0$. Let φ_n denote ε_n-approximate solution of (0.1) which exists on an interval $I = [t_0 - \eta, t_0 + \eta]$ where $\eta = \min\{a, \frac{b}{M}\}$ from Theorem 0.1 which was contructed as piecewise linear path as

$$\varphi_n(s) = \varphi_{n,i}(s) \quad \text{for} \quad s \in [t_{i-1}.t_i], \quad 1 \leq i \leq n \tag{0.16}$$

where

$$\begin{aligned} \varphi_{n,1}(s) &= l(s) = y_0 + f(t_0, y_0)(t - t_0) \quad \text{for} \quad t_0 \leq s \leq t_1 & (0.17) \\ \varphi_{2,n}(s) &= y_1 + f(t_1, y_1)(t - t_1) \quad \text{for} \quad s \in [t_1, t_2] \quad \text{where} \quad y_1 = \varphi_1(t_1) & (0.18) \end{aligned}$$

in general, denote $y_{i-1} = \varphi_{i-1}(t_{i-1})$ and define

$$\varphi_{n,i}(s) = y_{i-1} + f(t_{i-1}, y_{i-1})(t - t_{i-1}) \quad \text{for} \quad s \in [t_{i-1}, t_i] \quad \text{for} \quad i = 1, \ldots, n. \tag{0.19}$$

Here S_n is a finite set of points where the function φ_n is discontinuous. For all n, each φ_n satisfies

$$|\varphi_n(t) - \varphi_n(s)| < M|t - s| \quad \text{for all} \quad s, t \in I \tag{0.20}$$

where $M = \sup_R |f(t, y)|$, i.e., φ_n is Lipshcitz continuous on I. In particular, for $t \in I$,

$$\begin{aligned} |\varphi_n(t)| = |\varphi_n(t) - \varphi_n(t_0) + \varphi_n(t_0)| &\leq |\varphi_n(t) - \varphi_n(t_0)| + |\varphi_n(t_0)| \\ &< M|t - t_0| + |\varphi_n(t_0)| < M\eta + |\varphi_n(t_0)| \end{aligned}$$

hence $\{\varphi_n\}$ is uniformly bounded on I. Also, from (0.20) it follows that $\{\varphi_n\}$ is equicontinuous on I. From Ascoli's Lemma, it follows that $\{\varphi_n\}$ has a uniformly convergent subsequence on I, say $\{\varphi_{n_k}\}$ where φ_{n_k} is ε_{n_k}-approximate solution of (0.1). Let $\varphi_{n_k} \to \varphi$ uniformly on I. Then φ is continuous on I. Also,

$$\varphi_(t_0) = \lim_{n_k \to \infty} \varphi_{n_k}(t_0) = y_0.$$

To prove that φ satisfies the differential equation on I, we note that φ_{n_k} satisfies (0.1) on I except for finitely many points. Hence we can write

$$\varphi_{n_k}(t) = y_0 + \int_{t_0}^{t} \{f(s, \varphi_{n_k}(s)) + \Delta_{n_k}(s)\}\, ds$$

where

$$\begin{aligned} \Delta_n(s) &= \varphi'_{n_k}(s) - f(s, \varphi_{n_k}(s)) \quad \text{at} \quad \text{where } \varphi'_{n_k}(s) \text{ exists,} & (0.21) \\ &= 0 \quad \text{otherwise} & (0.22) \end{aligned}$$

Now taking limit as $n_k \to \infty$, $f(s, \varphi_{n_k}(s)) \to f(s, \varphi(s))$ and $\int_{t_0}^{t} \{f(s, \varphi_{n_k}(s))\, ds \to \int_{t_0}^{t} \{f(s, \varphi(s))\, ds$ since $\varphi_{n_k} \to \varphi$ uniformly on I. While since φ_{n_k} is an ε_{n_k} approximate solution

$$\int_{t_0}^{t} |\Delta_n(s)|\, ds = \int_{t_0}^{t} |\varphi'_{n_k}(s) - f(s, \varphi_{n_k}(s))|\, ds \leq \varepsilon_{n_k} |t - t_0| \to 0 \quad \text{as} \quad n_k \to \infty.$$

It follows that φ is solution of (0.1).

□