TYBSc Complex Analysis Notes

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1 Cauchy Goursat theorem

LEMMA 1.1 *Let $f : A \to \mathbb{C}$ be an analytic function such that $f'(z)$ is continuous in a domain $A \subset \mathbb{C}$, then $\int_C f(z)\,dz = 0$ for any simple closed curve $C \subset A$.*

Remark: The lemma is proved using Green's theorem, a result of Real analysis; for which we will use the identification of complex plane with $\mathbb{R}^2$, $x + iy \mapsto (x, y)$ so that a function on complex domain can be thought of as function of two real variables.

Proof:- Let $\gamma : [a, b] \to A$, $\gamma(t) = x(t) + iy(t)$ denote a parametrization of C and write $f(z) = u(x, y) + iv(x, y)$. Then

$$\begin{aligned}
\int_C f(z)\,dz &= \int_a^b f(\gamma(t)) \cdot \gamma'(t)\,dt \\
&= \int_a^b [u(\gamma(t)) + iv(\gamma(t))] \cdot (x'(t) + iy'(t))\,dt \\
&= \int_a^b \Big\{u(\gamma(t))x'(t) - v(\gamma(t))y'(t) + i[u(\gamma(t))y'(t) + v(\gamma(t))x'(t)]\Big\}dt \\
&= \int_a^b \Big\{u(\gamma(t))x'(t) - v(\gamma(t))y'(t)\Big\}dt + i\int_a^b \Big\{u(\gamma(t))y'(t) + v(\gamma(t))x'(t)\Big\}dt \\
&= \int_a^b \Big\{u(x(t) + iy(t))x'(t) - v(x(t) + iy(t))y'(t)\Big\}dt \\
&\quad + i\int_a^b \Big\{u(x(t) + iy(t))y'(t) + v(x(t) + iy(t))x'(t)\Big\}dt.
\end{aligned}$$

Therefore

$$\int_C f(z)dz = \int_C [u(x,y)dx - v(x,y)dy] + i\int_C [v(x,y)dx + u(x,y)dy].$$

By Greens' theorem, we have

$$\int_C [u(x,y)dx - v(x,y)dy] = \iint_R [-v_x - u_y]\, dxdy$$

and

$$\int_C [v(x,y)dx + u(x,y)dy] = \iint_R [u_x - v_y]\, dxdy$$

where R is the region enclosed by simple closed curve C. Hence

$$\int_C f(z)dz = \iint_R (-v_x - u_y)\, dxdy + i\iint_R (u_x - v_y)dxdy.$$

But since f is analytic in A, by C-R equations $u_x = v_y$ and $u_y = -v_x$. It follows that

$$\int_C f(z)\, dz = 0.$$

THEOREM 1.1 Cauchy Goursat theorem

Let $R \subset \mathbb{C}$ be a rectangle of length S and $f : R \to \mathbb{C}$ be an analytic function. Let D be a domain such that $\overline{D} \subset R$ and ∂D is a simple closed curve. Then $\int_{\partial D} f(z)dz = 0$

Proof:- Since f is analytic in R, given $\epsilon > 0$, there exists $\delta > 0$ such that $\left|\frac{f(z)-f(z_0)}{z-z_0} - f'(z_0)\right| < \epsilon$ whenever $|z - z_0| < \delta$. Cover $\overline{D}$ by rectangles $R(z)$ centered at $z = a + ib \in D$ defined as

$$R(z) := \{x + iy : |x - a| < \rho,\ |y - b| < \rho\}$$

where $\rho = \min\{\frac{\delta}{\sqrt{2}}, dist.(\partial D, \partial R)\}$. Note that $dist.(\partial D, \partial R) > 0$ (Why?)

Since $\overline{D} \subset R$ is compact, there exists finitely many points, say $z_1, z_2, \cdots, z_N \in D$ and rectangles R_j centered at z_j such that $\overline{D} \subset \bigcup_{j=1}^{N} R_j$. Moreover, for $z \in R_j$

$$\left|\frac{f(z) - f(z_j)}{z - z_j} - f'(z_j)\right| < \epsilon$$

Rewrite $\overline{D} = \left[\bigcup_{j\in I_1}(R_j \cap \overline{D})\right] \bigcup \left[\bigcup_{j\in I_2}(R_j \cap D)\right]$, where I_1 is index set for all rectangles $\overline{R}_j$ which lie in the interior of $\overline{D}$. The second union is over all rectangles with $j \in I_2$ for which the part of boundary $D_j = R_j \cap \partial D$ lies in the interior of R_j. Now for each j define,

$$\delta_j(z) = \frac{f(z) - f(z_j)}{z - z_j} - f'(z_j)$$

so that

$$f(z) = f(z_j) + f'(z_j)(z - z_j) + \delta_j(z)(z - z_j) \quad \text{in} \quad R_j.$$

Tracing the curve in anticlockwise direction, we see that

$$\int_{\partial D} f(z)dz = \sum_{j\in I_1} \int_{\partial R_j} f(z)dz + \sum_{j\in I_2} \int_{C_j \cup D_j} f(z)\,dz$$

where C_j denotes the part of boundary of R_j which lies in $\overline{D}$ for $j \in I_2$.

For $j \in I_1$,

$$\begin{aligned}
\int_{\partial R_j} f(z)dz &= \int_{\partial R_j} \Big[f(z_j) + f'(z_j)(z - z_j) + \delta_j(z)(z - z_j)\Big] dz \\
&= f(z_j) \int_{\partial R_j} dz + f'(z_j) \int_{\partial R_j} dz + \int_{\partial R_j} \delta_j(z)(z - z_j)dz \\
&= \int_{\partial R_j} \delta_j(z)(z - z_j)dz
\end{aligned}$$

since by Lemma 1.1 $\int_{\partial R_j} dz = 0 = \int_{\partial R_j}(z - z_j)dz$ (verify conditions of lemma). Therefore,

$$\begin{aligned}
\Big|\int_{\partial R_j} f(z)\Big| &= \Big|\int_{\partial R_j} \delta_j(z)(z - z_j)dz\Big| \\
&\le \int_{\partial R_j} |\delta_j(z)| \cdot |(z - z_j)| \cdot |dz| \\
&< \int_{\partial R_j} \epsilon \cdot |z - z_j| \cdot |dz| \\
&= \epsilon\sqrt{2}S_j \int_{\partial R_j} |dz| \\
&= \epsilon\sqrt{2}S_j \cdot 4S_j = 4\sqrt{2}\epsilon S_j^2 = 4\sqrt{2}\epsilon \cdot A(R_j)
\end{aligned}$$

where S_j is the length of R_j and $A(R_j)$ denotes the area of R_j. Similarly,

$$\begin{aligned}
\Big|\int\limits_{C_j\cup D_j} f(z)dz\Big| &\leq \int\limits_{C_j\cup D_j} |f(z)|\cdot|dz| \\
&\leq \int\limits_{C_j\cup D_j} |\delta_j(z)|\cdot|z-z_j|\cdot|dz| \\
&< \int\limits_{C_j\cup D_j} \epsilon\cdot|z-z_j|\cdot|dz| \\
&= \epsilon\sqrt{2}S_j \int\limits_{C_j\cup D_j} |dz| \\
&< \epsilon\sqrt{2}S_j(4S_j+l(D_j)) < \epsilon\sqrt{2}(4A(R_j)+S_jl(D_j)) < \epsilon\sqrt{2}(4A(R_j)+Sl(D_j))
\end{aligned}$$

where S is the length of R. Therefore

$$\begin{aligned}
\Big|\int\limits_{\partial D} f(z)dz\Big| &< \sum_{j\in I_1} 4\sqrt{2}\epsilon\cdot A(R_j) + \sum_{j\in I_2} 4\epsilon\sqrt{2}A(R_j) + \sum_{j\in I_2}\epsilon\sqrt{2}Sl(D_j) \\
&= 4\epsilon\sqrt{2}\sum_{j=1}^{N} A(R_j) + \epsilon\sqrt{2}Sl(\partial D) < \epsilon\sqrt{2}\left[4A(R)+Sl(\partial D)\right]
\end{aligned}$$

for any $\epsilon > 0$. It follows that $\Big|\int_{\partial D} f(z)\,dz\Big| = 0$ and $\int_{\partial D} f(z)\,dz = 0$.

2 Cauchy Integral formula and Taylor's theorem

THEOREM 2.1 Cauchy Integral Formula
Let A be an open connected subset of $\mathbb{C}$ and $f : A \to \mathbb{C}$ be an analytic function in A. Let $z_0 \in A$ and $r > 0$ such that $B(z_0, r) \subset A$. Then for any $w \in B(z_0, r)$ we have

$$f(w) = \frac{1}{2\pi i}\int\limits_{\partial B(z_0,r)} \frac{f(z)}{z-w}\,dz. \tag{2.1}$$

Proof:
Step I: We first prove $\int_{\partial B(z_0,r)} \frac{1}{z-w}\,dz = 2\pi i$:
For given $z_0 \in A$ and $w \in B(z_0, r)$, let $r_1 > 0$ such that $B(w, r_1) \subset B(z_0, r)$. Write

$z = w + r_1 e^{i\theta}$, then $dz = ir_1 e^{i\theta}$. Thus

$$\int_{\partial B(w,r_1)} \frac{1}{z-w}\, dz = \int_0^{2\pi} \frac{1}{r_1 e^{i\theta}} \cdot r_1 i e^{i\theta}\, d\theta = i \int_0^{2\pi} d\theta = 2\pi i.$$

Next we show that

$$\int_{\partial B(z_0,r)} \frac{1}{z-w}\, dz = \int_{\partial B(w,r_1)} \frac{1}{z-w}\, dz.$$

Let AB, CD be a straight line segments joining points A and D on the circle $\partial B(w, r_1)$ and the points B and C respectively, on the circle $\partial B(z_0, r)$. Consider the closed curve C_1 traced in anticlockwise direction along segment $\overrightarrow{AB}$, the arc of the circle $\partial B(z_0, r)$ from B to C given by $\Gamma(BC) := \{z = z_0 + re^{i\theta} : 0 \leq \theta \leq \pi\}$, along the segment $\overrightarrow{CD}$ and the circle $\partial B(w, r_1)$ from D to A given by $\Gamma(DA) := \{z = z_0 + r_1 e^{i\theta} : \pi \leq \theta \leq 2\pi\}$. For $z \in C_1$, $z - w$ is non zero in the interior of region, say R_1, bounded by C_1, and hence $\frac{1}{z-w}$ is analytic in R_1. By Cauchy-Goursat theorem

$$\int_{C_1} \frac{1}{z-w}\, dz = 0$$

i.e.,

$$\int_A^B \frac{1}{z-w}\, dz + \int_{\Gamma(BC)} \frac{1}{z-w}\, dz + \int_C^D \frac{1}{z-w}\, dz + \int_{\Gamma(DA)} \frac{1}{z-w}\, dz = 0. \tag{2.2}$$

Similarly, considering the closed curve C_2 traced in anticlockwise direction along the arc of the circle $\partial B(z_0, r_1)$ from A to D given by $\Gamma(AD) := \{z = z_0 + r_1 e^{i\theta} : 0 \leq \theta \leq \pi\}$, along the segment $\overrightarrow{DC}$, the circle $\partial B(w, r)$ from C to B given by $\Gamma(CB) := \{z = z_0 + re^{i\theta} : \pi \leq \theta \leq 2\pi\}$ and finally the segment $\overrightarrow{BA}$, we have for $z \in C_2$, since $z - w$ is non zero in the interior of region, say R_2, bounded by C_2,

$$\int_{\Gamma(AD)} \frac{1}{z-w}\, dz + \int_D^C \frac{1}{z-w}\, dz + \int_{\Gamma(CB)} \frac{1}{z-w}\, dz + \int_B^A \frac{1}{z-w}\, dz = 0. \tag{2.3}$$

Adding (2.2) and (2.3), it follows that $\int_{\partial B(z_0,r)} \frac{1}{z-w}\, dz = \int_{\partial B(w,r_1)} \frac{1}{z-w}\, dz$.

Step II: Due to Step I, (2.1) can be rewritten as

$$\int_{\partial B(z_0,r)} \frac{f(z) - f(w)}{z-w} dz = 0.$$

Since $f : A \to \mathbb{C}$ is analytic in A, for given $\varepsilon > 0$ there exists $\delta > 0$ such that

$$\left| \frac{f(z) - f(w)}{w - z} - f'(w) \right| < \varepsilon \quad \text{whenever} \quad |z - w| < \delta$$

Choose $r_1 < \delta$, then

$$\int\limits_{\partial B(w,r_1)} \frac{f(z) - f(w)}{z - w}\, dz = \int\limits_{\partial B(w,r_1)} \left(\frac{f(z) - f(w)}{z - w} - f'(w) \right) + \int\limits_{\partial B(w,r_1)} f'(w)\, dz.$$

But by Cauchy Goursat's theorem, $\int_{\partial B(w,r_1)} f'(w) dz = 0$ since constant function is analytic in $B(w, r_1)$. Hence

$$\left| \int\limits_{\partial B(w,r_1)} \frac{f(z) - f(w)}{z - w}\, dz \right| \leq \int\limits_{\partial B(w,r_1)} \left| \frac{f(z) - f(w)}{z - w} - f'(w) \right| |dz| < \varepsilon \pi r_1 < \varepsilon \pi r.$$

Since $\varepsilon > 0$ can be chosen arbitrarily small, we conclude

$$\int\limits_{\partial B(w,r_1)} \frac{f(z) - f(w)}{z - w}\, dz = 0.$$

Again defining a closed curve as in Step I by joining $\partial B(z_0, r)$ with $\partial B(w, r_1)$ by a line segment, since the function $\frac{f(z)-f(w)}{z-w}$ is analytic in the region $B(z_0, r) \setminus B(w, r_1)$, applying Cauchy Goursat's theorem we have

$$\int_{\partial B(z_0,r)} \frac{f(z) - f(w)}{z - w}\, dz = \int_{\partial B(w,r_1)} \frac{f(z) - f(w)}{z - w}\, dz = 0$$

which completes the proof of the theorem.

THEOREM 2.2 *If $f : \Omega \to \mathbb{C}$ is analytic in Ω then its derivative of all orders are analytic in Ω. For all $w \in B(z_0, r) \subset \Omega$,*

$$f^{(n)}(w) = \frac{n!}{2\pi i} \int\limits_{\partial B(z_0,r)} \frac{f(z)}{(z - w)^{n+1}}\, dz.$$

Moreover, if $M(r)$ is the maximum value of $|f(z)|$ on $\partial B(z_0, r)$ then

$$|f^{(n)}(w)| \leq \frac{n! M(r)}{r^n}.$$

Proof:

THEOREM 2.3 Taylor's theorem: *Suppose that f is analytic in a disc $B(z_0, R_0)$. Then $f(z)$ has the power series representation*

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n$$

where

$$a_n = \frac{f^n(z_0)}{n!} = \frac{1}{2\pi i} \int\limits_{\partial B(z_0, r)} \frac{f(z)}{(z - z_0)^{n+1}} \, dz$$

Proof:

3 Integration and differentiation of power series of complex numbers

THEOREM 3.1 *Let C be a simple closed curve in the interior of the disc of convergence of the power series $S(z) = \sum\limits_{n=0}^{\infty} a_n(z - z_0)^n$ and let $g(z)$ be any function which is continuous on C. Then the series $\sum\limits_{n=0}^{\infty} g(z) a_n (z - z_0)^n$ can be integrated term by term over C and*

$$\int\limits_C g(z) S(z) \, dz = \sum_{n=0}^{\infty} \int\limits_C g(z) a_n (z - z_0)^n \, dz.$$

i.e., a complex power series can be integrated term by term over C within its disc of convergence.

• A function is g is said to be continuous on C is for any parametrization $\gamma : [a, b] \to \mathbb{C}$ of C, the composition $g \circ \gamma : [a, b] \to \mathbb{C}$ is continuous.

• Observe that $g(z)S(z) = \sum\limits_{n=0}^{\infty} g(z) a_n (z - z_0)^n$.

• Note that the result is more general than the one we prove for real power series. This result will be used in the following theorem which proves term by term differentiation of power series within its disc of convergence.

THEOREM 3.2 *Let C be a simple closed curve in the interior of the disc of convergence of the power series* $S(z) = \sum_{n=0}^{\infty} a_n(z-z_0)^n$,

$$S'(z) = \sum_{n=1}^{\infty} na_n(z-z_0)^{n-1}.$$

The above equality says that the sum of differentiated series is derivative of the sum. The following proof is different from one given in Brown and Churchill.

Proof: **Step I:** We will first show that $S(z)$ satisfies the Cauchy integral formula. Let $s_n(z) = \sum_{k=0}^{n} a_k(z-z_0)^k$ denote the sequence of partial sums of $S(z)$. Since the function s_n is analytic in $B(z_0, R_0)$, by Cauchy integral formula, for $w \in B(z_0, R)$ we have

$$s_n(w) = \frac{1}{2\pi i} \int_C \frac{s_n(z)}{(z-w)} \, dz$$

for a simple closed curve $C = \partial B(w, r)$, $B(w, r) \subset B(z_0, R_0)$. Applying Theorem 3.1 to the r.h.s. with $g(z) = \frac{1}{(z-w)}$ which is continous on C we have

$$\lim_{n\to\infty} \frac{1}{2\pi i} \int_C \frac{s_n(z)}{(z-w)} \, dz = \frac{1}{2\pi i} \int_C \frac{S(z)}{(z-w)} \, dz$$

while $\lim_{n\to\infty} s_n(w) = S(w)$. Thus

$$S(w) = \frac{1}{2\pi i} \int_C \frac{S(z)}{(z-w)} \, dz.$$

Step II: We have

$$\begin{aligned} S'(w) &= \lim_{h\to 0} \frac{S(w+h) - S(w)}{h} \\ &= \lim_{h\to 0} \frac{1}{2\pi i} \int_C S(z) \frac{1}{h} \left(\frac{1}{(z-w-h)} - \frac{1}{(z-w)} \right) dz \\ &= \frac{1}{2\pi i} \int_C S(z) \frac{1}{(z-w)^2} \, dz \\ &= \lim_{n\to\infty} \frac{1}{2\pi i} \int_C s_n(z) \frac{1}{(z-w)^2} \, dz \quad \text{from Theorem 3.1} \\ &= \lim_{n\to\infty} s_n'(z) \quad \text{since } s_n(z) \text{ is analytic} \\ &= \lim_{n\to\infty} \sum_{k=1}^{n} ka_k(z-z_0)^{k-1}. \end{aligned}$$

This shows that the series $\sum k a_k (z-z_0)^{k-1}$ converges and its sum is $S'(z)$.

COROLLARY 3.1 *$S(z)$ is analytic in $B(z_0, R_0)$.*

THEOREM 3.3 Uniqueness of series representation:
If a series $\sum a_n (z-z_0)^n$ converges to $f(z)$ at all ponts within the disc of convergence $|z-z_0| < R$ then it is the Taylor series expansion for f centered at z_0.

Remark: We have proved that analytic function has a power series representation (Taylor's theorem) and above shown that if a function can be represented by a power series, then it is analytic within its disc of convergence. Complex differentiation implies existence of Taylor series, which is different from "real differentiation".

4 Singularities

• Isolated singular point: A point z_0 is called an isolated singular point of a map $f : \Omega \to \mathbb{C}$ if there exists a deleted neighbourhood of z_0 say, $0 < \zeta - z_0| < \delta \subset \Omega$ in which f is analytic.

• Define double series of complex numbers $\sum_{n=-\infty}^{\infty} a_n$, its convergence and double power series.

• Laurent's theorem says that in neighbourhood os an isolated singular point, the function has Laurent series expansion. It will be used to classify the isolated singular points:

THEOREM 4.1 Laurent's theorem *Suppose that a function f is analytic throughout an annular domain $A := \{z \in \mathbb{C} : R_1 < |z - z_0| < R_2\}$ centered at z_0 and $C \subset A$ be a positively oriented simple closed curve around z_0. Then for each $z \in A$, $f(z)$ has the Laurent series representation*

$$f(z) = \sum_{n=0}^{\infty} a_n (z-z_0)^n + \sum_{n=1}^{\infty} \frac{b_n}{(z-z_0)^n}$$

where

$$\begin{aligned} a_n &= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{n+1}}\, dz, \quad n = 0, 1, 2, 3, \dots \\ b_n &= \frac{1}{2\pi i} \int_C \frac{f(z)}{(z-z_0)^{-n+1}}\, dz, \quad n = 1, 2, 3, \dots . \end{aligned}$$

The point z_0 is said to be

• **a removable singularity** if $b_n = 0$ for all $n = 1, 2, 3, \ldots$.

• **a pole** if there exists some m such that $b_n \neq 0$ for all $n = 1, 2, \ldots, m$ and $b_n = 0$ for all $n = m+1, m+2, m+3, \ldots$. In this case, z_0 is said to be **a pole of order** m.

• **an essential singularity** if $b_n \neq 0$ for infinitely many n.

• The constant b_1 is defined to be **the residue of f at the singular point** z_0 denoted by $Resf(z_0)$ which is given by

$$Resf(z_0) = b_1 = \frac{1}{2\pi i} \int_C f(z)\, dz.$$