## 1. Fundamental theorem of space curves

**THEOREM 1.1.** *Given differentiable functions* $\kappa(s) > 0$ *and* $\tau(s)$, $s \in (a,b) \subseteq I\!R$, *there exists a regular parametrized curve* $\gamma : (a,b) \to I\!R^3$ *such that* $s$ *is the arc length,* $\kappa(s)$ *is the curvature and* $\tau(s)$ *is the torsion of* $\gamma$.

*Moreover, if* $\tilde{\gamma}$ *is any other curve satisfying the same conditions, then there exists a rigid motion ( an isometry ) of* $I$ *of* $I\!R^3$ *such that*

$$\tilde{\gamma} = I(\gamma)$$

*i.e.,* $\tilde{\gamma}(s) = \rho \circ \gamma(s) + a$ *where* $\rho$ *is a rotation and* $a \in I\!R^3$ *is some fixed vector.*

*Proof.* Step 1: To construct a regular parametrized curve $\gamma$ with curvature $\kappa$ and torsion $\tau$: The Serret- Frenet equations for space curves parametrised by arc length are

$$\begin{aligned} \overrightarrow{t}'(s) &= \kappa(s)\overrightarrow{n}(s) \\ \overrightarrow{n}'(s) &= -\kappa(s)\overrightarrow{t}(s) - \tau(s)\overrightarrow{b}(s) \\ \overrightarrow{b}'(s) &= \tau(s)\overrightarrow{n}(s) \end{aligned}$$

where $\overrightarrow{t}(s)$ is the unit tangent vector, $\overrightarrow{n}(s)$ is a unit normal vector and $\overrightarrow{b}(s)$ is unit binormal vector in $I\!R^3$. Thus, for the given functions $\kappa$ and $\tau$ consider the system of ordinary differential equations

$$\left.\begin{aligned} T'(s) &= \kappa(s)N(s) \\ N'(s) &= -\kappa(s)T(s) - \tau(s)B(s) \\ B'(s) &= \tau(s)N(s) \end{aligned}\right\} \tag{1.1}$$

and we look for solution $F : (a,b) \to I\!R^9$, $F(s) = (T(s), N(s), B(s))$ with the initial condition

$$F(0) = (e_1, e_2, e_3)$$

where $e_1 = (1,0,0)$, $e_2 = (0,1,0)$ and $e_3 = (0,0,1)$ is the standard basis in $I\!R^3$. In terms of $F$ the system (1.1) can be written as

$$\left.\begin{aligned} \tfrac{dF}{ds} &= A(s)F(s) \\ F(s_0) &= (1,0,0,01,0,0,0,1) \end{aligned}\right\} \tag{1.2}$$

where $A(s)$ is a $9 \times 9$ matrix

$$A(s) = \begin{pmatrix} O_{3\times3} & \kappa(s)Id_{3\times3} & O_{3\times3} \\ -\kappa(s)Id_{3\times3} & O_{3\times3} & -\tau(s)Id_{3\times3} \\ O_{3\times3} & \tau(s)Id_{3\times3} & O_{3\times3} \end{pmatrix} \tag{1.3}$$

where

$$O_{3\times3} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

is the $3 \times 3$ zero matrix and

$$Id_{3\times 3} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

is the $3 \times 3$ identity matrix.

The existence and uniqueness theorem of ODE states: *Given a map $X : (a, b) \times U \to I\!R^n$ which is Lipschitz continuous on an open set $U \subset I\!R^n$ and a point $(s_0, \hat{x}_1, \ldots, \hat{x}_n) \in (a, b) \times U$, there exists a unique solution $\alpha : (s_0 - \delta, s_0 + \delta) \to U$, $\delta > 0$ to the initial value problem*

$$\begin{aligned} \alpha'(s) &= X(s, \alpha(s)) \\ \alpha(s_0) &= (\hat{x}_1, \ldots, \hat{x}_n) \end{aligned}$$

*Moreover, if the map $X(s, .)$ is linear (in this case $U = I\!R^n$) then the solution $\alpha(s)$ is defined for all $s \in (a, b)$.*

Since in (1.2) the map $X(s, v) = A(s)v$ for $(s, v) \in (a, b) \times I\!R^9$ is linear, there exists a unique solution $F : (a, b) \to I\!R^9$ to the initial value problem (1.2).

We claim that the vectors $\{T(s), N(s), B(s)\}$ are orthonormal, for each $s \in (a, b)$. To see this we define the functions

$$\begin{aligned} \sigma_1(s) &= < T(s), T(s) > \\ \sigma_2(s) &= < N(s), N(s) > \\ \sigma_3(s) &= < B(s), B(s) > \\ \eta_1(s) &= < T(s), N(s) > \\ \eta_2(s) &= < N(s), B(s) > \\ \eta_3(s) &= < B(s), T(s) > \end{aligned}$$

and write the differential equations satisfied by them using (1.1). Observe that

$$\sigma_1'(s) = 2 < T(s), T'(s) >= 2 < T(s), \kappa(s)N(s) >= 2\kappa(s) < T(s), N(s) >= 2\kappa(s)\eta_1(s)$$

using the equation (1.2). Similar computations can be done for remaining functions using (1.2) to get

$$\begin{aligned} \sigma_1'(s) &= 2\kappa(s)\eta_1(s) \\ \sigma_2'(s) &= -2\kappa(s)\eta_1(s) - 2\tau(s)\eta_2(s) \\ \sigma_3'(s) &= 2\tau(s)\eta_2(s) \\ \eta_1'(s) &= \kappa(s)\sigma_2(s) - \kappa(s)\sigma_1(s) - \tau(s)\eta_3(s) \\ \eta_2'(s) &= -\kappa(s)\eta_3(s) - \tau(s)\sigma_3(s) + \tau(s)\sigma_2(s) \\ \eta_3'(s) &= \tau(s)\eta_1(s) + \kappa(s)\eta_2(s) \end{aligned}$$

Thus $G(s) = (\sigma_1(s), \sigma_2(s), \sigma_3(s), \eta_1(s), \eta_2(s), \eta_3(s)) \in I\!R^6$ is a solution for the system

$$\left.\begin{array}{rcl} \frac{dG}{ds} & = & B(s)G(s) \\ G(s_0) & = & (1,1,1,0,0,0) \end{array}\right\} \tag{1.4}$$

where $B(s)$ is a $6 \times 6$ matrix

$$B(s) = \begin{pmatrix} 0 & 0 & 0 & 2\kappa(s) & 0 & 0 \\ 0 & 0 & 0 & -2\kappa(s) & 0 & -\tau(s) \\ 0 & 0 & 0 & 0 & 2\tau(s) & 0 \\ -\kappa(s) & \kappa(s) & 0 & 0 & 0 & -\tau(s) \\ 0 & \tau(s) & -\tau(s) & 0 & 0 & -\kappa(s) \\ 0 & 0 & 0 & \tau(s) & \kappa(s) & 0 \end{pmatrix}. \tag{1.5}$$

Since the system is linear and the constant curve $\alpha(s) = (1,1,1,0,0,0)$ is a solution of the system (1.4), uniqueness of solution implies that $G(s) = \alpha(s)$ for all $s$ and hence the claim is true.

Define $\gamma(s) := \int\limits_{s_0}^{s} T(s)\, ds$.

**Exercise:** Verify that $\gamma : (a,b) \to I\!R^3$ is the required curve.

Step II: Let $\tilde{\gamma} : (a,b) \to I\!R^3$ be a regular parametrized curve with curvature $\kappa(s)$ and torsion $\tau(s)$. For $s_1 \in (a,b)$, let $p = \gamma(s_1)$ be a point on the curve represented by $\gamma$ and $\tilde{p} = \tilde{\gamma}(s_1)$ be the point on $\tilde{\gamma}$. The vectors $\{\overrightarrow{t}(s_1), \overrightarrow{n}(s_1), \overrightarrow{b}(s_1)\}$ and $\{\overrightarrow{\tilde{t}}(s_1), \overrightarrow{\tilde{n}}(s_1), \overrightarrow{\tilde{b}}(s_1)\}$ are both orthonormal bases of $I\!R^3$ (similar to the standard basis in $I\!R^3$). Hence there exists a rotation $\rho$ of $I\!R^3$ which maps the orthonormal basis $\{\overrightarrow{t}(s_1), \overrightarrow{n}(s_1), \overrightarrow{b}(s_1)\}$ onto $\{\overrightarrow{\tilde{t}}(s_1), \overrightarrow{\tilde{n}}(s_1), \overrightarrow{\tilde{b}}(s_1)\}$. Choose a vector $a \in I\!R^3$ such that $T_a(\gamma(s_1)) = \gamma(s_1) + a = \tilde{\gamma}(s_1)$, i.e., $T_a$ translates the point $p$ to $\tilde{p}$.

Define $I = T_a \circ \rho$ and let $\gamma_1(s) := I(\gamma(s)$. Note that $I$ is an isometry of $I\!R^3$.

**Ex:** Show that curvature of $\gamma_1$ is $\kappa(s)$ and the torsion is $\tau(s)$.

Clearly, both $\gamma_1$ and $\gamma$ are solutions of the initial value problem

$$\left.\begin{array}{rcl} \frac{dF}{ds} & = & A(s)F(s) \\ F(s_1) & = & p \end{array}\right\} \tag{1.6}$$

with $A(s)$ as in (1.3).Thus existence and uniqueness theorem implies that

$$\gamma(s) = \gamma_1(s)$$

for all $s \in (a,b)$.

$\square$